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# EQUATIONS FOR A SEMIEMPIRCAL THEORY OF TURBULENT TRANSPORT

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ABSTRACT: A closed system of equations for the anisotropic nature of a flow is constructed on the assumption that the length of the mixing length is not short compared with the characteristic dimension of the flow. It is taken that the velocity pulsation field can be characterized by a multipoint distribution function satisfying the equation of continuity. This results in equations for single-point and double-point distribution functions.

A number of suggestions are advanced dealing with the nature of the forces acting on the turbulent formation ("mole" or vortex) in the flow with the correlation time between the random force and the scale and the intensity of the turbulence, with the expression of the integral in the equation for the single-point distribution function, and with the expression for the correlation tensor in the isotropic case. The computation of moments yields a system of Reynolds equations in which approximations usually taken from considerations of dimensionality are made for a number of summands. Here this is the result of approximations for the forces in the distribution function equation. Closure of the system of equations for the moments reduces to solving the equation for the distribution function. And it is shown that the integral nature of the transport (direct diffusion) is associated with consideration of third order moments. A number of examples of flows fixing the values of the empirical constants are reviewed. A system of equations is obtained for use in considering a flow with strong anisotropy of turbulent transport. /62*

Generally known results, based on the inclusion of equations for second moments and considerations of dimensionality for the expression for the

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* Numbers in the margin indicate pagination in the foreign text.

corresponding summands in the equation for the turbulent energy balance in terms of intensity and turbulence scale, are contained in the investigations made by A. N. Kolmogorov, L. Prandtl, J. Rotta, and others (a bibliography and summary of these works can be found in [1]).

A number of papers have attempted to describe the turbulent transport process by using kinetic equations [2], as well as by using an analogy to the neutron transport and radiation processes [3,4]. Reference [4] lists the corresponding bibliography.

1. Turbulent transport model. It is assumed that a multipoint distribution function, which can characterize the velocity pulsation field, $f^{(N)}(q_i, p_i, T, \chi)$ $(i = 1, \ldots, N)$, where $q_i$ are coordinates, $p_i$ are formation pulses, T is the temperature, $\chi$ is the admixture concentration, satisfies the equation of continuity.

$$\frac{\partial f^{(N)}}{\partial t} + \frac{\partial}{\partial q_i}\left[\frac{p_i}{m} f^{(N)}\right] + \frac{\partial}{\partial p_i}\left[\frac{dp_i}{dt} f^{(N)}\right] + \frac{\partial}{\partial T}\left[\frac{dT}{dt} f^{(N)}\right] + \frac{\partial}{\partial \chi}\left[\frac{d\chi}{dt} f^{(N)}\right] = 0 \quad (1.1)$$

(only a flow with small changes in temperature and concentrations will be considered, such that the magnitudes with density pulsation should be ignored everywhere except for the summand with the acceleration of gravity [1]).

The simplest assumption possible is made with respect to the formation scales: at every point in the flow the size of the moles can be characterized by a single scale value proportional to the integral correlation scale with which the magnitude of the mixing length is associated, and defined as the distance characterizing the loss of correlation between the original and final positions of the mole [5].

The simultaneous distribution function at n points /63

$$f^{(n)} = \int f^{(N)} \prod_{n+1}^{N} dp_m \, dq_m$$

satisfies the equation

$$\frac{\partial f^{(n)}}{\partial t} + \frac{\partial}{\partial q_i}\left[\frac{p_i}{m} f^{(n)}\right] + \int_{1 \leqslant i \leqslant n} \frac{\partial}{\partial p_i}\left[\frac{dp_i}{dt} f^{(N)}\right] \prod_{n+1}^{N} dp_m \, dq_m + \frac{\partial}{\partial T}\left[\frac{dT}{dt} f^{(n)}\right] + \frac{\partial}{\partial \chi}\left[\frac{d\chi}{dt} f^{(n)}\right] = 0 \quad (1.2)$$

One of the basic assumptions concerns the form of the expression for the force acting on the mole. It is assumed that this force comprises two parts: the first, $F_i = (dp_i / dt)_1$, describes the hydrodynamic interaction of the mole with the flow, thanks to the existence of the relative velocity, and is in form similar to that for the force acting on a sphere with radius L; the second is connected with the action of fluctuations in pressure on the mole. This is a random force that depends on all the coordinates in the field. Ye. A. Novikov used random forces to describe the turbulent flow field in [6]. It was taken that fluctuations in pressure change quite rapidly compared with change in the distribution function, and that the force associated with their action has correlation time $\tau$. Then integration of Eq. (1.2) in the correlation time limits from $-\tau$ to 0 yields [7].

$$\begin{gathered}\frac{\partial f^{(n)}}{\partial t}+\frac{\partial}{\partial q_i}\left[\frac{p_i}{m}f^{(n)}\right]+\frac{\partial}{\partial p_i}\left[\left(\frac{dp_i}{dt}\right)_1 f^{(n)}\right]+\frac{\partial}{\partial T}\left[\frac{dT}{dt}f^{(n)}\right]+\\+\frac{\partial}{\partial\chi}\left[\frac{d\chi}{dt}f^{(n)}\right]=\frac{1}{\tau}\left[-f^{(n)}+\int f^{(n)}(\mathbf{p}_s-\Delta\mathbf{p}_s)W(\Delta\mathbf{p}_s)\Pi d(\Delta\mathbf{p}_s)\right]\end{gathered}\tag{1.3}$$

Here $W(\Delta p)$ is the probability of pulse deviation by $\Delta p$. Since there is no possibility of obtaining an exact solution by using Eq. (1.3), an attempt will be made to use it to obtain expressions for single-point and two-point distribution functions in terms of the parameters for an inhomogeneous flow field.

Let us assume that $f^{(1)} = f$ for $n = 1$, and

$$\begin{gathered}\frac{\partial f}{\partial t}+\frac{\partial}{\partial q_i}\left[\frac{p_i}{m}f\right]+\frac{\partial}{\partial p_i}\left[\left(\frac{dp_i}{dt}\right)_1 f\right]+\frac{\partial}{\partial T}\left[\frac{dT}{dt}f\right]+\frac{\partial}{\partial\chi}\left[\frac{d\chi}{dt}f\right]=\\=\frac{1}{\tau}\left[-f+\int f(\mathbf{p}-\Delta\mathbf{p})W(\Delta\mathbf{p})d(\Delta\mathbf{p})\right]\end{gathered}\tag{1.4}$$

The following approximation is taken for the expression for the integral in the right side

$$\begin{gathered}\int f(\mathbf{p}-\Delta\mathbf{p})W(\Delta\mathbf{p})d(\Delta\mathbf{p})\doteq f_0\\ f_0=\langle\rho\rangle\left(\frac{4\pi E}{3}\right)^{-3/2}\exp\left[-\frac{3(u_i-\langle u_i\rangle)^2}{4E}\right](\pi\langle T'^2\rangle)^{-1/2}\exp\left[-\frac{(T-\langle T\rangle)^2}{\langle T'^2\rangle}\right]\times\\ \times(\pi\langle\chi'^2\rangle)^{-1/2}\exp\left[-\frac{(\chi-\langle\chi\rangle)^2}{\langle\chi'^2\rangle}\right],\qquad E=\frac{1}{2}\langle u_k'^2\rangle\end{gathered}$$

Here $f_o$ is the local equilibrium value of the distribution function. The considerations for this approximation are purely qualitative, and are based on the fact that this approximation is widely used in the kinetic theory of gases, although the sense of the right side is different, as well as in what is completely an equilibrium case. /64

The relationship used on the basis of dimensionality considerations is

$$\tau = ALE^{-1/2} \tag{1.5}$$

where A is an empirical constant, for $\tau$ magnitudes, and this is customary. This simplifies Eq. (1.4) considerably

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial q_i}\left[\frac{p_i}{m} f\right] + \frac{\partial}{\partial p_i}\left[\left(\frac{dp_i}{dt}\right)_1 f\right] + \frac{\partial}{\partial T}\left[\frac{dT}{dt} f\right] + \frac{\partial}{\partial \chi}\left[\frac{d\chi}{dt} f\right] = (f_0 - f)/\tau \tag{1.6}$$

Certain relationships based on experimental data concerning the friction, heat exchange, and mass exchange processes for a body moving in a fluid must be taken for the magnitudes $F_i = (dp_i / dt)_1$, $dT/dt$, and $d\chi/dt$. In particular

$$F_i = -\frac{1}{\rho}\frac{\partial \langle P\rangle}{\partial x_i} - g\delta(x_i, z) - |C|\, u_i' \frac{3a_0}{8L} \tag{1.7}$$

Here the resisting force is approximated by an expression that is valid for large values of the local Reynolds number (see [3,8] as well).

It would be desirable to retain the integral nature of the solution for the distribution functions f and $f^{(2)}$ in toto, but this can only be done for the function f. The developed construction can be improved if success can be had in obtaining a more satisfactory presentation for f and $f^{(2)}$.

2. Equations. Multiplying Eq. (1.6) by $Q(u, T, \chi)$, and integrating over the entire variable space u, T, $\chi$, we obtain the transport equation

$$\frac{\partial}{\partial t}\langle\rho\rangle\langle Q\rangle + \frac{\partial}{\partial x_i}\langle\rho\rangle\langle u_i Q\rangle - \langle\rho\rangle\left\langle F_i \frac{\partial Q}{\partial u_i}\right\rangle - \langle\rho\rangle\left\langle \frac{dT}{dt}\frac{\partial Q}{\partial T}\right\rangle - \\ - \langle\rho\rangle\left\langle \frac{d\chi}{dt}\frac{\partial Q}{\partial \chi}\right\rangle = \int \tau^{-1} Q\,(f_0 - f)\, du\, dT\, d\chi \tag{2.1}$$

Assuming $Q = 1$, $u_\alpha$, $u_\alpha^2$, $u_\alpha u_\beta$, $T$, $u_\alpha T$, $T^2$, $\chi$, $u_\alpha \chi$, $\chi^2$, we can obtain equations for the mean magnitudes and second moments (summing not done in terms of $\alpha$ and $\beta$). Then it is taken that the magnitude $\langle\rho\rangle$ is a constant. Strictly speaking, this corresponds to constancy in flow for the value of the energy of the pulsation motion. When the entire flow is considered, including the regions near the solid wall, or the free boundary, where this condition is violated, it becomes necessary to introduce the magnitude "for the density of the turbulent state of the flow" $\langle\rho\rangle$, which is associated with the coefficient of intermittence, and what then must be considered is a flow consisting of two conditions for the medium: laminar, and turbulent.

The equation of continuity is

$$\frac{\partial \langle u_k \rangle}{\partial x_k} = 0 \tag{2.2}$$

The equation of motion for the mean velocity values is

$$\frac{\partial}{\partial t}\langle u_i \rangle + \frac{\partial}{\partial x_k}[\langle u_k \rangle \langle u_i \rangle + \langle u_i' u_k' \rangle] = -\frac{1}{\langle\rho\rangle}\frac{\partial \langle P \rangle}{\partial x_i} - g\delta(x_i, z) \tag{2.3}$$

The equations for the components of the tensor of the Reynolds stresses are

$$\begin{gathered}\frac{\partial}{\partial t}\langle u_\alpha' u_\beta' \rangle + \frac{\partial}{\partial x_k}[\langle u_k \rangle \langle u_\alpha' u_\beta' \rangle + \langle u_k' u_\alpha' u_\beta' \rangle] + \langle u_\beta' u_k' \rangle \frac{\partial \langle u_\alpha \rangle}{\partial x_k} + \\ + \langle u_\alpha' u_k' \rangle \frac{\partial \langle u_\beta \rangle}{\partial x_k} = -g\frac{\langle \rho' u_\beta' \rangle}{\langle\rho\rangle}\delta(x_\alpha, z) - g\frac{\langle \rho' u_\alpha' \rangle}{\langle\rho\rangle}\delta(x_\beta, z) - \\ - \langle |C| u_\alpha' u_\beta' \rangle 3a_0/4L + \tau^{-1}[{}^2/{}_3 E\delta(\alpha, \beta) - \langle u_\alpha' u_\beta' \rangle]\end{gathered} \tag{2.4}$$

The energy equation is

$$\frac{\partial C_p \langle T \rangle}{\partial t} + \frac{\partial}{\partial x_k}[C_p \langle T \rangle \langle u_k \rangle + C_p \langle T' u_k' \rangle] = \frac{1}{\langle\rho\rangle}\frac{d\langle P \rangle}{dt} \tag{2.5}$$

The equations for the components of the vector for the turbulent heat flux are

$$\begin{gathered}\frac{\partial}{\partial t}C_p \langle T' u_i' \rangle + \langle u_i' u_k' \rangle \frac{\partial C_p \langle T \rangle}{\partial x_k} + \frac{\partial}{\partial x_k}[\langle u_k \rangle \langle C_p T' u_i' \rangle] + \\ + C_p \langle T' u_k' \rangle \frac{\partial \langle u_i \rangle}{\partial x_k} + \frac{\partial}{\partial x_k} C_p \langle T' u_i' u_k' \rangle = -C_p \langle T' u_i' |C| \rangle \frac{3a_0}{8L} - \\ - \frac{1}{\langle\rho\rangle}\frac{d\langle P \rangle}{dt}\frac{\langle \rho' u_i' \rangle}{\langle\rho\rangle} - \tau^{-1}\langle C_p T' u_i' \rangle - C_p \langle T' \rho' \rangle g\delta(x_i, z)\end{gathered} \tag{2.6}$$

The equation for the intensity of temperature fluctuations is

$$\begin{gathered}\frac{\partial}{\partial t}\frac{C_p \langle T'^2 \rangle}{2} + \frac{C_p}{2}\frac{\partial}{\partial x_k}[\langle u_k \rangle \langle T'^2 \rangle + \langle T'^2 u_k' \rangle] + \\ + C_p \langle T' u_k' \rangle \frac{\partial \langle T \rangle}{\partial x_k} = -\frac{1}{\langle\rho\rangle}\frac{d\langle P \rangle}{dt}\frac{\langle \rho' T' \rangle}{\langle\rho\rangle}\end{gathered} \tag{2.7}$$

No equations will be written for the concentration field. Eqs. (2.2)-(2.7), (1.6), (1.5), and the scale equation (cited in the next section) comprise a closed system. The third order moments contained in the second order moments equation must be determined by using the solution for the distribution function.

Let us take the expression $\sim E^{1/2}\langle u_\alpha' u_\beta' \rangle$ for the magnitude $\langle |C| u_\alpha' u_\beta' \rangle$.

3. Equation for scale L. An approximate equation for L can be obtained by integrating and angle averaging the equation for the second rank correlation tensor, just as Rotta [9] did. The only difference is that the original equation and the approximate expressions for the third order moments in this paper are based on Eq. (1.3) for the distribution function $f^{(2)}$.

This equation is multiplied by $u_\alpha^{(a)}$ and $u_\beta^{(b)}$, and integrated over the entire variable space. The result of using the equation of motion at points a and b, and the equation of continuity, is to obtain the transport equation in the following form. Let us introduce the following variables: the distance between points a and b of the flow, $\zeta$, and the coordinates $x_k^{(ab)}$.

$$\zeta_k = x_k^{(b)} - x_k^{(a)}, \quad x_k^{(ab)} = {}^1/_2\,[x_k^{(b)} + x_k^{(a)}]$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\langle u_\alpha'^{(a)} u_\beta'^{(b)}\rangle + \langle u_k'^{(a)} u_\beta'^{(b)}\rangle \left[\frac{\partial\langle u_\alpha\rangle}{\partial x_k}\right]^{(a)} + \langle u_\alpha'^{(a)} u_k'^{(b)}\rangle \left[\frac{\partial\langle u_\beta\rangle}{\partial x_k}\right]^{(b)} + \\
&+ \frac{1}{2}[\langle u_k^{(a)}\rangle + \langle u_k^{(b)}\rangle]\frac{\partial}{\partial x_k^{(ab)}}\langle u_\alpha'^{(a)} u_\beta'^{(b)}\rangle + [\langle u_k^{(b)}\rangle - \langle u_k^{(a)}\rangle]\frac{\partial}{\partial \zeta_k}\langle u_\alpha'^{(a)} u_\beta'^{(b)}\rangle + \\
&+ \frac{1}{2}\frac{\partial}{\partial x_k^{(ab)}}[\langle u_\alpha'^{(a)} u_\beta'^{(b)} u_k'^{(b)}\rangle + \langle u_\alpha'^{(b)} u_k'^{(a)} u_\beta'^{(b)}\rangle] + \\
&+ \frac{\partial}{\partial \zeta_k}[\langle u_\alpha'^{(a)} u_k'^{(b)} u_\beta'^{(b)}\rangle - \langle u_\alpha'^{(a)} u_k'^{(a)} u_\beta'^{(b)}\rangle] = -g\delta(x_\alpha, z)\frac{\langle \rho'^{(a)} u_\beta'^{(b)}\rangle}{\langle \rho\rangle} - \\
&- g\delta(x_\beta, z)\frac{\langle \rho'^{(b)} u_\alpha'^{(a)}\rangle}{\langle \rho\rangle} - \langle |C|^{(a)} u_\alpha'^{(a)} u_\beta'^{(b)}\rangle \frac{3a_0}{8L_{(a)}} - \\
&- \langle |C|^{(b)} u_\alpha'^{(a)} u_\beta'^{(b)}\rangle \frac{3a_0}{8L_{(b)}} + \int u_\alpha^{(a)} u_\beta^{(b)} \tau^{-1}(F - f^{(2)})\, du^{(a)}\, du^{(b)}\, dT\, d\chi
\end{aligned} \qquad (3.1)$$

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As will be seen, the scale is not a scalar magnitude in the general case. But it is desirable to give further consideration to the simplest case during the first approximatation consideration, when a number of simplifying assumptions are made (one of which is the introduction of but one mean scale at a point). We will, therefore, proceed as follows. We write the equation for the sum $\langle u_i'^{(a)} u_i'^{(b)}\rangle$, integrate the equation at the point (ab) with respect to the distance $\zeta$ between points a and b, and with respect to the angle, and take the mean value

$${}^1/_2\,[\langle u_k^{(a)}\rangle + \langle u_k^{(b)}\rangle] \approx \langle u_k^{(ab)}\rangle, \qquad L_{(a)} \approx L_{(b)} \approx L_{(ab)}$$

It is assumed that the latter summand, which depends on the pulsations in pressure, is in the form $\sim \tau^{-1}\langle u'^{(a)}_\alpha u'^{(b)}_\beta\rangle$, and that it can be combined with the summands corresponding to the dissipation. Moreover, we will assume that

$$\int \langle u_i'^{(a)} u_i'^{(b)}\rangle d\zeta\, d\Omega \sim \langle u_i'^2\rangle L = 2LE$$

$$\int \langle u_k'^{(a)} u_l'^{(b)}\rangle d\zeta\, d\Omega \sim 2L\langle u_k' u_l'\rangle \xi_0$$

The result is the following approximate equation for L

$$\frac{\partial}{\partial t}(LE) + \xi_0 L\langle u_i' u_k'\rangle \frac{\partial\langle u_i\rangle}{\partial x_k} + \langle u_k\rangle \frac{\partial}{\partial x_k}(LE) = \\ = -\frac{1}{2}\frac{\partial}{\partial x_k}\left[\int \left(\langle u_i'^{(a)} u_i'^{(b)} u_k'^{(b)}\rangle + \langle u_i'^{(a)} u_i'^{(b)} u_k'^{(a)}\rangle\right)\frac{d\zeta\, d\Omega}{4\pi}\right] - \\ - \alpha_s E^{3/2} 3a_0/4 - 2\alpha_g g L\langle \rho' u_z'\rangle / \langle\rho\rangle \tag{3.2}$$

Let us consider the following, very simple, model for purposes of determining the form of the dependency of the third order moments for the inhomogeneous case. The principal role in the equation for $f^{(2)}$ is played by pulsations in pressure, and the resisting forces can be ignored. Then the case of the stationary field can be considered, for which the equation for $f^{(2)}$ in coordinates $\zeta_k$, $x_k^{(ab)}$ has the form

$$1/2\,(u_k^{(a)} + u_k^{(b)})\,\partial f^{(2)}/\partial x_k^{(ab)} + (u_k^{(a)} - u_k^{(b)})\,\partial f^{(2)}/\partial\zeta_k = \tau^{-1}(F - f^{(2)}) \tag{3.3}$$

In the homogeneous case $f_o^{(2)}$ is dependent only on $\zeta$ and satisfies the equation

$$(u_k^{(a)} - u_k^{(b)})\,\partial f_0^{(2)}/\partial\zeta_k = \tau^{-1}(F - f_0^{(2)})$$

We obtain the correction for the expression for the homogeneous case by substituting the value $f_o^{(2)}$ from Eq. (3.3) in the expression for $f^{(2)}$

$$f^{(2)} = F - \tau(u_k^{(a)} - u_k^{(b)})\,\partial f^{(2)}/\partial\zeta_k - 1/2\,\tau(u_k^{(a)} + u_k^{(b)})\,\partial f^{(2)}/\partial x_k^{(ab)} \approx \\ \approx f_0^{(2)} - 1/2\,\tau(u_k^{(a)} + u_k^{(b)})\,\partial f_0^{(2)}/\partial x_k^{(ab)} \tag{3.4}$$

In this expression the parameters E, L, and $\langle u_m\rangle$ must be considered as functions of the coordinates. The following derivatives can be contained in the right side during differentiation /67

$$\partial E/\partial x_k,\quad \partial L/\partial x_k,\quad u_m \partial\langle u_m\rangle/\partial x_k$$

Assuming equality to zero of the "semiinvariants" (cumulants) [1] of fourth and fifth orders, the fourth moments can be expressed in terms of the second, and the fifth, which prove to be proportional to the third, should be ignored in the approximation considered for weak inhomogeneity. Specifically

$$\begin{aligned}\langle u_i'^{(a)} u_i'^{(b)} u_k'^{(b)} \rangle \approx \langle u_i'^{(a)} u_i'^{(b)} u_k'^{(b)} \rangle_0 - \\ - \sum [\gamma_1' L E^{-1/2} \langle u_i'^{(a)} u_i'^{(b)} u_k'^{(b)} u_l'^{(\ )} \rangle \partial E / \partial x_l + \\ + \gamma_2' E^{-1/2} \langle u_i'^{(a)} u_i'^{(b)} u_k'^{(b)} u_l'^{(\ )} \rangle \partial L / \partial x_l + \gamma_3' L E^{-1/2} \langle u_i'^{(a)} u_i'^{(b)} u_k'^{(b)} u_m'^{(ab)} u_l'^{(\ )} \rangle \times \\ \times \frac{\partial \langle u_m' \rangle}{\partial x_l} \Big] \approx \langle u_i'^{(a)} u_i'^{(b)} u_k'^{(b)} \rangle_0 - \sum E^{-1/2} [\langle u_i'^{(a)} u_i'^{(b)} \rangle \langle u_k'^{(b)} u_l'^{(\ )} \rangle + \\ + \langle u_i'^{(a)} u_k'^{(b)} \rangle \langle u_i'^{(b)} u_l'^{(\ )} \rangle + \langle u_i'^{(a)} u_l'^{(\ )} \rangle \langle u_i'^{(b)} u_k'^{(b)} \rangle ] \times \\ \times [\gamma_1' L \partial E / \partial x_l + \gamma_2' E \partial L / \partial x_l]\end{aligned}$$

After integration with respect to the distance between the points and to the angle, the result is

$$\begin{aligned}\int \langle u_i'^{(a)} u_i'^{(b)} u_k'^{(b)} \rangle \frac{d\zeta d\Omega}{4\pi} \approx \\ \approx - L E^{1/2} \left[ \frac{\langle u_k' u_l' \rangle}{E} + \frac{\langle u_i' u_k' \rangle \langle u_i' u_l' \rangle}{E^2} \right] \left[ \gamma_1 L \frac{\partial E}{\partial x_l} + \gamma_2 E \frac{\partial L}{\partial x_l} \right]\end{aligned} \tag{3.5}$$

The magnitudes of the constant $\gamma_1$ and $\gamma_2$ should be determined from the experimental data. Substituting Eq. (3.5) in Eq. (3.2), we obtain the final equation for the scale as

$$\begin{aligned}\frac{\partial}{\partial t} LE + \xi_0 L \langle u_i' u_k' \rangle \frac{\partial \langle u_i \rangle}{\partial x_k} + \langle u_k \rangle \frac{\partial}{\partial x_k} LE = \\ = \frac{\partial}{\partial x_k} \Big\{ L E^{1/2} \left[ \frac{\langle u_k' u_l' \rangle}{E} + \frac{\langle u_i' u_k' \rangle \langle u_i' u_l' \rangle}{E^2} \right] \\ [\gamma_1 L \partial E / \partial x_l + \gamma_2 E \partial L / \partial x_l] \Big\} - \alpha_s E^{3/2} 3 a_0 / 4 - 2 \alpha_g g L \langle \rho' u_z' \rangle / \langle \rho \rangle\end{aligned} \tag{3.6}$$

This equation is in keeping with the scale equation obtained by Rotta [9], who used a number of relationships and hypotheses for spectral functions in its development. Here the equation is obtained by using distribution functions.

4. The equations in the foregoing refer to a developed turbulent flow, and do not take into consideration the processes associated with

molecular diffusion. Consideration of the flow in the region where viscosity has a substantial effect requires correct computation of third moments, because these latter play a determinant role in this region.

The basis for the determination of the magnitudes of the third moments should be the solution of the equation for the distribution function. There is no possibility of structuring a solution for f in simple form, so we structure an approximate solution that will properly consider the fundamental characteristic of the turbulent transport process; the generation of moles, and their propagation over considerable distances. The effect of the re- /68
sistive forces will be ignored in the equation of the distribution function, although the respective summands are not small, generally speaking, and we will consider a plane flow in which the flow parameters depend solely on the transverse coordinate. Then

$$u_y' \frac{\partial f}{\partial y} = \tau^{-1}(f_0 - f)$$
$$f(y, u_y' < 0) = \int_y^{\infty} f_0 \exp\left[-\int_y^{s} \frac{ds}{\tau|u_y'|}\right]\frac{ds}{\tau|u_y'|} \qquad (4.1)$$
$$f(y, u_y' > 0) = \int_{-\infty}^{y} f_0 \exp\left[-\int_s^{y} \frac{ds}{\tau u_y'}\right]\frac{ds}{\tau u_y'}$$

The equation of viscous shear is in the form

$$\frac{d}{dy}\langle u_x' u_y'^2\rangle + \langle u_y'^2\rangle \frac{d\langle u_x\rangle}{dy} = -\tau^{-1}\langle u_x' u_y'\rangle \qquad (4.2)$$

Computing the magnitudes

$$\langle u_x' u_y'\rangle = \int (u_x - \langle u_x\rangle)\, u_y f\, du_x du_y du_z$$
$$\langle u_y'^2\rangle = \int u_y^2 f\, du_x du_y du_z \qquad (4.3)$$
$$\langle u_x' u_y'^2\rangle = \int (u_x - \langle u_x\rangle)\, u_y^2 f\, du_x du_y du_z$$

we can be persuaded that Eq. (4.2) is satisfied once a direct substitution is made. But viscous shear Eq. (4.3) describes the integral transport when the mixing length is long, so diffusion of the direct type is equivalent to the computation for the third order moment.

Let us here note further that approximations for forces acting on the mole and adopted in the distribution function equation led to expressions that were used for the summands containing pressure fluctuations, and were introduced because of considerations of dimensionality and because of physical considerations. There is a difference in the summand characterizing the dissipation of energy, for here the isotropic expression

$$\tfrac{3}{4}a_0 L^{-1} E^{3/2} \delta(\alpha, \beta)$$

usually was used.

And the following expression too was obtained here

$$\tfrac{3}{4}a_0 L^{-1} E^{1/2} \langle u_\alpha' u_\beta' \rangle$$

that is, the dissipation of energy for the component of the kinetic energy of the pulsation motion, $\langle u_x'^2 \rangle$ for example, is proportional to this magnitude $\sim E^{1/2}\langle u_x'^2 \rangle$, and cannot be taken as equal to one-third of the total magnitude of dissipation of energy.

5. The steady plane flow of a turbulent flow. Let us consider a steady isothermic plane flow, again for the case when the flow parameters change in the direction of the y axis only. Flow velocity, U, is directed along the x axis.

Layer of constant viscous shear. This flow can be observed at a distance /69
from the wall (viscosity effect can be ignored). Third order moments equal zero. Let us introduce the dimensionless magnitudes

$$U^+ = U v_*^{-1}, \quad y^+ = y v_* \nu^{-1}$$

We obtain as a result $(B = 1 + \tfrac{3}{4}a_0 A)$

$$\langle u_x' u_y' \rangle v_*^{-2} = -1$$

$$-2\frac{dU^+}{dy^+} = -\frac{B\langle u_x'^2\rangle^+ (E^+)^{1/2}}{AL^+} + \tfrac{2}{3}\frac{(E^+)^{3/2}}{AL^+}$$

$$\frac{B\langle u_y'^2\rangle^+ (E^+)^{1/2}}{AL^+} = \tfrac{2}{3}\frac{(E^+)^{3/2}}{AL^+}$$

$$\frac{dU^+}{dy^+} = \tfrac{3}{4}\frac{a_0 (E^+)^{3/2}}{L^+}, \qquad \langle u_y'^2\rangle^+ \frac{dU^+}{dy^+} = \frac{B(E^+)^{1/2}}{AL^+}$$

$$-\xi_1 L^+ \frac{dU^+}{dy^+} = \gamma_2 \frac{d}{dy^+}\left\{L^+ (E^+)^{3/2}\left[\frac{\langle u_y'^2\rangle^+}{E^+}\left(1 + \frac{\langle u_y'^2\rangle^+}{E^+}\right) + (E^+)^{-2}\right]\frac{dL^+}{dy^+}\right\} - \frac{3}{4}\alpha_s a_0 (E^+)^{3/2}$$

These equations yield

$$L^+ \frac{dU^+}{dy^+} = \frac{3a_0 (E^+)^{3/2}}{4}, \quad \frac{\langle u_x'^2\rangle^+}{E^+} = \frac{2(3B-2)}{3B}$$

$$\frac{\langle u_y'^2\rangle^+}{E^+} = \frac{2}{3B}, \quad E^+ = B\left[\frac{2}{a_0 A}\right]^{1/2}$$

$$\frac{d}{dy^+}\left(L^+ \frac{dL^+}{dy^+}\right) = \text{const} = \beta_0^2$$

What follows from this latter is

$$L^+ = \beta_0 y^+$$

and from the first

$$U^+ = C_0 + \frac{3a_0 B^{3/2}}{4\beta_0}\left(\frac{2}{a_0 A}\right)^{3/4} \ln y^+ = C_0 + \varkappa^{-1} \ln y^+$$

that is, the logarithmic law for velocity distribution known for this case. The value of the constant $\varkappa = 0.40$. We will also say that $\beta_o = 0.40$, and that $a_o = 0.345$. Then $a_o A = 4/3$ and $A = 3.86$. The relationships given above will yield the following for these values of the constants

$$E^+ = 2{,}44 \quad \langle u_x'^2\rangle = 3{,}24, \quad \langle u_y'^2\rangle = 0{,}81$$

These values are in satisfactory concordance with the experimental data [10].

Isotropic plane flow beyond a lattice. The equations are greatly simplified for isotropic flow because

$$\langle u_x'^2\rangle = \langle u_y'^2\rangle = \langle u_z'^2\rangle = {}^2/_3 E$$

Viscous forces can be ignored, velocity U is constant, and we have the expressions

$$u\frac{dE}{dx} = -\frac{E^{3/2} 3a_0}{4L}; \quad u\frac{dLE}{dx} = -\frac{\alpha_s E^{3/2} 3a_0}{4}$$

Third order moments too have been ignored here. What follows from these equations is the relationship for the L. G. Loytsyanskiy invariant [11] /70

$$\Lambda = EL^{1/(1-\alpha_s)}$$

and since $1 - \alpha_s$ should equal $1/5$, $\alpha_s = 0.8$.

It is now easy to find the dependencies

$$L \sim x^{2/7}, \quad E \sim x^{-10/7}$$

This corresponds to A. N. Kolmogorov's known results.

Flow in a wind tunnel. Here it is taken that scale L is constant, that is is proportional to the dimensions of the cells in the lattice, whereupon the equation [12] that follows is valid

$$\frac{dE}{dt} = -\frac{3}{2}\left(\frac{2}{3}\right)^{3/2} \frac{A''}{l} E^{3/2}, \qquad A'' = 1.0 \div 1.2$$

Here $l$ is the longitudinal integral scale, that is, a magnitude that is proportional to L (close to AL, because this value is included in the index of the exponent). In this case the equation of the magnitude E yields

$$dE\,/\,dt = -\,{}^{3}/_{4}\,a_0 L^{-1} E^{3/2}$$

Comparing these two relationships, we obtain the evaluation

$$l \approx 4L\,/\,3a_0 = AL$$

which agrees with the value $3/4a_oA = 1$, taken previously.

6. Passage to the limit relationships of the mixing length theory. A system of equations describes turbulent transport without the introduction of an assumption as to the smallness of the magnitude of the mixing length, L. L. Prandtl developed the mixing length theory on the basis of an analogy with molecular transport in a continuous medium mode when L is small compared with the characteristic size of the problem. We will come to the model of the mixing length theory if we take into consideration the fact that when $L \to 0$ the magnitudes $\tau$ and $\langle u_i' u_j' \rangle$ $(i \neq j)$ are small first order magnitudes, and that E is a finite magnitude. In order to satisfy these conditions, it must be taken that $a_o \to 0$. All these conditions are contradictory because in a turbulent flow the magnitude $\langle u_x' u_y' \rangle$ is not small compared with E, for example.

In the light of the conditions listed, the original equations (without the viscosity computation) yield

$$\begin{gathered}\langle u_\alpha'^2\rangle = {}^2/_3 E - 2\nu'\partial\langle u_\alpha\rangle/\partial x_\alpha\\ \langle u_\alpha' u_\beta'\rangle = -\nu'(\partial\langle u_\alpha\rangle/\partial x_\beta + \partial\langle u_\beta\rangle/\partial x_\alpha)\\ c_p\langle T' u_i'\rangle = -k'\partial c_p\langle T\rangle/\partial x_i\\ \nu' = k' = {}^2/_3 ALE^{1/2}, \quad P' = \nu'/k' = 1\end{gathered}$$

And the expression for the virtual viscosity becomes scalar.

If the passage to the limit, $L \to 0$, is not completed, we will obtain an expression for the Prandtl turbulence number, P', different from one. Let us consider the special case of plane-parallel motion (without taking into consideration viscous forces and the third order moment)

$$\begin{gathered}\langle u_x' u_y'\rangle = -\frac{AL}{(1+3a_0A/4)E^{1/2}}\langle u_y'^2\rangle\frac{du}{dy}\\ c_p\langle T' u_y'\rangle = -\frac{AL}{(1+3a_0A/8)E^{1/2}}\langle u_y'^2\rangle\frac{dc_p\langle T\rangle}{dy}\end{gathered}$$

and for this case we have /71

$$P' = \frac{1 + {}^3/_8 a_0 A}{1 + {}^3/_4 a_0 A} = {}^3/_4$$

for the values of constants adopted.

Contained in the equation are the constants $a_o$, A, $\alpha_s$, $\xi_o$, $\gamma_1$, $\gamma_2$, from which the values of $\alpha_s$ and A can be determined from the relationships that follow from the equations, as we saw above. The others can be determined by comparing them with the experimental data.

As will be seen from the very simple examples used, the system of equations obtained makes it possible to describe known behavior patterns, permits consideration of direct diffusion and the nonisotropic behavior of the components of the kinetic energy of pulsation motion, and the introduction of nonisotropic eddy viscosity, which proves to be significant in many problems concerned with turbulent flow in a stratified medium.

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